HOSSAM GHANEM

(23) 3.7 Implicit Differentiation (A)

Example 1 — 29 July 25th, 2002

Let $x = \dfrac{x + xy + 2}{y^2}$ Find $\dfrac{dy}{dx}$ at $y = 1$

Solution

$$x = \frac{x + xy + 2}{y^2}$$

$$1 = \frac{y^2(1 + y + xy^\backslash) - (x + xy + 2)2yy^\backslash}{y^4}$$

at $y = 1$

$$x = \frac{x + x + 2}{1}$$

$$x = 2x + 2$$

$$x = -2$$

$$1 = \frac{(2 - 2y^\backslash) - (-2 - 2 + 2)\,2y^\backslash}{1}$$

$$1 = 2 - 2y^\backslash + 4y^\backslash$$

$$2y^\backslash = -1$$

$$y^\backslash = -\frac{1}{2}$$

Example 2 — 40 May 3, 2007

Use implicit differential to find $y^{\backslash\backslash}$ if $x^2 + y^4 = 16$

Solution

$$x^2 + y^4 = 16$$

$$2x + 4y^3\, y^\backslash = 0$$

$$y^\backslash = \frac{-2x}{4y^3}$$

$$y^{\backslash\backslash} = \frac{4y^3(-2) + 2x \cdot 12y^2 y^\backslash}{16\, y^6}$$

$$y^{\backslash\backslash} = \frac{-8\, y^3 + 24\, x\, y^2 \cdot \frac{-2x}{4y^3}}{16\, y^6} = \frac{-8\, y^3 + 24\, x \cdot \frac{-2x}{4y}}{16\, y^6} = \frac{-8\, y^3 - 12xy^{-1}}{16\, y^6}$$

Example 3

Use implicit differential to find $\dfrac{dy}{dx}$ if $\dfrac{1}{x} + \dfrac{1}{y} = 1$

Solution

$$\frac{1}{x} + \frac{1}{y} = 1$$

$$\frac{-1}{x^2} - \frac{1}{y^2}\, y^{\backslash} = 0$$

$$-\frac{1}{y^2}\, y^{\backslash} = \frac{1}{x^2}$$

$$y^{\backslash} = -\frac{y^2}{x^2}$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2}$$

Example 4

Use implicit differential to find $\dfrac{dy}{dx}$ if $3xy = (x^3 + y^2)^{\frac{3}{2}}$

Solution

$$3xy = (x^3 + y^2)^{\frac{3}{2}}$$

$$3y + 3xy^{\backslash} = \frac{3}{2}(x^3 + y^2)^{\frac{1}{2}}\,(3x^2 + 2yy^{\backslash})$$

$$3y + 3xy^{\backslash} = \frac{9}{2}x^2(x^2 + y^2)^{\frac{1}{2}} + 3y(x^3 + y^2)^{\frac{1}{2}}\, y^{\backslash}$$

$$3xy^{\backslash} - 3y(x^3 + y^2)^{\frac{1}{2}}\, y^{\backslash} = \frac{9}{2}x^2(x^2 + y^2)^{\frac{1}{2}} - 3y$$

$$y^{\backslash}\left[3x - 3y(x^3 + y^2)^{\frac{1}{2}}\right] = \frac{9}{2}x^2(x^2 + y^2)^{\frac{1}{2}} - 3y$$

$$y^{\backslash} = \left[\frac{9}{2}x^2(x^2 + y^2)^{\frac{1}{2}} - 3y\right]\left[3x - 3y(x^3 + y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

Example 5	35 December 16, 2004 Find equations of the normal lines to the graph of equation $4x + y^2 + (xy)^2 = 6$ at the point whose x-coordinate is 1

Solution

$4x + y^2 + (xy)^2 = 6$

at $x = 1$

$4 + y^2 + y^2 = 6 \quad \Rightarrow \quad 2y^2 = 2 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow y = \pm 1$

$p_1\ (1, -1) \quad , \quad p_2\ (1,1)$

$4 + 2yy^{\backslash} + 2xy^2 + 2x^2yy^{\backslash} = 0$

at $x = 1 \ , \quad y = -1$

$4 - 2y^{\backslash} + 2 - 2y^{\backslash} = 0$

$-4y^{\backslash} + 6 = 0$

$y^{\backslash} = \dfrac{3}{2}$

$p_1\ (1, -1) \quad , \quad m = -\dfrac{2}{3}$

$y - y_1 = m\ (x - x_1)$

$y + 1 = -\dfrac{2}{3}(x - 1)$

at $x = 1 \ , \quad y = -1$

$4 + 2y^{\backslash} + 2 + 2y^{\backslash} = 0$

$4y^{\backslash} = -6$

$y^{\backslash} = \dfrac{-3}{2}$

$p_2\ (1,1) \quad , \quad m = \dfrac{2}{3}$

$y - y_1 = m\ (x - x_1)$

$y - 1 = \dfrac{2}{3}(x - 1)$

Example 6	Show that the tangent lines to the graph of $3x - 2y + x^3 + x^2y = 0$ and $x^2 - 2x + y^2 - 3y = 0$ at the origin are perpendicular to each other

Solution

$3x - 2y + x^3 + x^2y = 0$

$3 - 2y^{\backslash} + 3x^2 + 2xy + x^2\,y^{\backslash} = 0$

at $(0,0)$

$3 - 2y^{\backslash} = 0 \quad \Rightarrow \quad y^{\backslash} = \dfrac{3}{2} \quad \Rightarrow \quad m_1 = \dfrac{3}{2}$

$x^2 - 2x + y^2 - 3y = 0$

$2x - 2 + 2yy^{\backslash} - 3y^{\backslash} = 0$

at $(0,0)$

$-2 - 3y^{\backslash} = 0 \quad \Rightarrow \quad y^{\backslash} = \dfrac{-2}{3} \quad \Rightarrow \quad m_2 = \dfrac{-2}{3}$

$m_1 \cdot m_2 = \dfrac{3}{2} \cdot \dfrac{-2}{3} = -1$

$\therefore\ L_1$ is perpendicular to L_2

Example 7	25 December 10, 2000 Find the point (s) at which the tangent line to the graph of $3x^2 - 2y^2 = 1$ is perpendicular to the line $2x + 3y + 1 = 0$

Solution

$L_1: \; 2x + 3y + 1 = 0 \qquad \Rightarrow m_1 = \frac{-2}{3}$

$3x^2 - 2y^2 = 1 \qquad \rightarrow (1)$

$6x^2 - 4yy' = 1 \qquad \Rightarrow y' = \frac{6x^2}{4y} \qquad \Rightarrow m_2 = \frac{3x^2}{2y}$

$m_1 \cdot m_2 = \frac{-2}{3} \cdot \frac{3x^2}{2y} = -1$

$\frac{x^2}{y} = 1 \qquad \Rightarrow x^2 = y \qquad \rightarrow (2)$

from (2) in (1)

$3y - 2y^2 = 1$

$2y^2 - 3y + 1 = 0 \qquad \Rightarrow (2y-1)(y-1) = 0$

$y = \frac{1}{2} \quad \text{or} \quad y = 1$

$x = \pm\frac{1}{\sqrt{2}} \quad or \quad x = \pm 1$

the points : $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), (-1,1), (1,1)$

Example 8	47 December 22, 2009 Find an equation for the tangent line to the graph of $(x+y)^2 = 1 + x^2y^2$ at the point whose x –coordinate is 0

Solution

$(x+y)^2 = 1 + x^2y^2$

$x = 0 \qquad \Rightarrow y^2 = 1 \qquad \Rightarrow y = \pm 1$

the points : $p_1(0,-1)$, $\quad p_2(0,1)$

$2(x+y)(1+y') = 2xy^2 + 2x^2yy'$

at $p_1(0,-1)$

$2(-1)(1+y') = 0 \qquad \Rightarrow 1 + y' = 0$

$y' = -1 \qquad \Rightarrow m_1 = -1$

$p_1(0,-1) \qquad m_1 = -1$

$y - y_1 = m(x - x_1)$

$y + 1 = -x$

$\therefore \; y + 1 + x = 0$

at $p_2(0,1)$

$2(1)(1+y') = 0 \qquad \Rightarrow 1 + y' = 0$

$y' = -1$

$p_2(0,1) \qquad m = -1$

$y - y_1 = m(x - x_1) \qquad \Rightarrow y - 1 = -x$

$\therefore \; y + 1 - x = 0$

Example 9

Find the points on the graph of equation 18 December 3, 1998

$x^2 - 2xy + 2y^2 = 4$ at which the tangent line is horizontal

Solution

$x^2 - 2xy + 2y^2 = 4 \quad \rightarrow (1)$

$2x - 2y - 2xy^{\backslash} + 4yy^{\backslash} = 0$

$y^{\backslash}(4y - 2x) = 2y - 2x$

$y^{\backslash} = \dfrac{2y - 2x}{4y - 2x}$

$H.T \quad at \quad 2y - 2x = 0$

$x = y \quad (2) \rightarrow$

$from \ (2) \ 1n \ (1)$

$x^2 - 2x^2 + 2x^2 = 4$

$x^2 = 4 \quad \Rightarrow x = \pm 2 \quad \Rightarrow y = \pm 2$

$the\ points \ : p_1(-2, -2) \ , \quad p_2(2, 2)$

Homework

	Find $y^\backslash$ Find $\frac{dy}{dx}$	
1	$x^2 + y^2 = 4$	
2	$(x - y)^2 - y = 0$	
3	$(y + 3x)^2 - 4x = 0$	
4	$(x^2 + 3y^2)^{35} = x$	
5	$x^3 + y^3 - 3xy = 0$	
6	$xy + (x + y)^3 = 1$	26 May 10, 2001
7	Find $\frac{dy}{dx}$ at $x = 0$ if $xy + (x + y)^3 = 1$	

1	Find the equation of the tangent line to the graph of $5x^2 + 4y^2 = 56$ at the point $(-2\,,\ 3)$
2	16 July 30, 1998 Find the equation of the tangent line to the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at the point $P(\,0\,,-2\,)$
3	17 August 4, 1998 Find the equation of the normal line to the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the point $P(\,2\,,0\,)$
4	41 July 19, 2007 Show that the curves $x^2 + y^2 = r^2$ and $y = mx$ are orthogonal (meet at right angles) for all constants $r \neq 0\,, m \neq 0$
5	Find the point(s) at which the tangent line to the graph of $3x^2 - 2y^2 = 1$ is perpendicular to the line $2x + 3y + 1 = 0$
6	Find all the point on the graph of $x^2y^2 + xy = 2$ where the slope of the tangent line is -1
7	23 April 27,2000 Let f and g be functions for which $f^\backslash(x) = g(x)$ and $g^\backslash(x) = f(x)$ for all x . Show that $f^2(x) - g^2(x)$ is a constant function
8	27 August 2, 2001 Find an equation of the normal line to the curve $x^3 + 2x^2y + y^2 = 1$ at the point $P(\,1\,,0\,)$

Homework

9	6 January 6, 1993 Let y be a function of x implicitly defined by $3x^2 - xy^2 + 2y = 12$ Find the equation of normal line to the graph of y at the point $(2, 0)$
10	9 January 9, 1994 Let y be a function of x defined implicitly by $xy(xy + x + y) = 1$ Find the equation of the normal line to the graph of y at $x = 1$
11	1 November 1987 Assuming that the equation $x^3y^3 + yx^2 + y + xy^2 = 3$ determines implicitly a differentiable function f such that $y = f(x)$. Find an equation for the normal to the curve at $(0, 3)$
12	4 May 19, 1992 Find the equation of the tangent line to the graph of $x^3 + y^3 - 4xy = 0$ at the point $(2, 2)$
13	Assuming that the equation $x^3y^3 + yx^2 + y + xy^2 = 5$ determines implicitly a differentiable function f such that $y = f(x)$. Find an equation for the normal to the curve at $(0, 5)$
14	5 July 13, 1992 Find an equation of the normal line to the graph of $y^3 + xy^2 + x^2 + 3 = 0$ at the point $(1, -2)$
15	30 May 15th, 2003 Find an equation of the normal line to the graph of $xy + (x + y)^3 + 1 = 0$ at $x = 0$
16	14 December 18, 1997 Find the equation of the tangent line to the graph of $(x^2 + y^2)^2 = 4xy$ at the point $P(1, 1)$
17	Use implicit differential to find $\frac{dy}{dx}$ if $x^2 = \frac{x + y}{x - y}$
18	Use implicit differential to find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = 4$
19	23 April 27,2000 Find $\frac{dy}{dt}$ at $t = 9$ given that $y = \frac{u + 2}{u - 1}$ $u = \left(3\sqrt{t} - 7\right)^2$
20	Find the equation of the normal line to the graph of the equation $\sqrt{x} + \sqrt{y} = 3$ at the point where $x = 1$

Homework

21	Find the points, if any ,on the graph of $y^2 - 6x^2 + 4x + 1 = 0$ at which the tangent lines have slop 4

21	Find the points, if any ,on the graph of $y^2 - 6x^2 + 4x + 1 = 0$ at which the tangent lines have slop 4

Solution

$y^2 - 6x^2 + 4x + 1 = 0 \qquad \to (1)$

$2yy' - 12x + 4 = 0$

$y' = \dfrac{12x - 4}{2y}$

$m = \dfrac{6x - 2}{y}$

$\dfrac{6x - 2}{y} = 4$

$y = \dfrac{6x - 2}{4} = \dfrac{3x - 1}{2} \qquad \to (2)$

(2)in (1)

$\left(\dfrac{3x - 1}{2}\right)^2 - 6x^2 + 4x + 1 = 0$

$\dfrac{9x^2 - 6x + 1}{4} - 6x^2 + 16x + 4 = 0$

$9x^2 - 6x + 1 - 24x^2 + 16x + 4 = 0$

$-15x^2 + 10x + 5 = 0$

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$x = -\dfrac{1}{3} \qquad or \qquad x = 1$

$y = \dfrac{-1 - 1}{2} = -1 \qquad or \qquad y = \dfrac{3 - 1}{2} = 1$

The points $\left(-\frac{1}{3}, -1\right)$ & $(1,,1)$